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New perspectives on the relevance of gravitation for the covariant description of electromagnetically polarizable media

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Abstract

By recognizing that stress–energy–momentum tensors are fundamentally related to gravitation in spacetime it is argued that the classical electromagnetic properties of a simple polarizable medium may be parameterized in terms of a constitutive tensor whose properties can in principle be determined by experiments in non-inertial (accelerating) frames and in the presence of weak but variable gravitational fields. After establishing some geometric notation, discussion is given to basic concepts of stress, energy and momentum *in the vacuum* where the useful notion of a *drive* form is introduced in order to associate the conservation of currents involving the flux of energy, momentum and angular momentum with spacetime isometries. The definition of the stress–energy–momentum tensor is discussed with particular reference to its symmetry based on its role as a source of relativistic gravitation. General constitutive properties of *material continua* are formulated in terms of spacetime tensors including those that describe *magneto-electric phenomena* in moving media. This leads to a formulation of a self-adjoint constitutive tensor describing, in general, inhomogeneous, anisotropic, magneto-electric bulk matter in arbitrary motion. The question of an invariant characterization of intrinsically magneto-electric media is explored. An action principle is established to generate the phenomenological Maxwell system and the use of variational derivatives to calculate stress–energy–momentum tensors is discussed in some detail. The relation of this result to tensors proposed by Abraham and others is discussed in the concluding section where the relevance of the whole approach to experiments on matter in non-inertial environments with variable gravitational and electromagnetic fields is stressed.

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1. Introduction

The laws of quantum-electrodynamics have been devised to describe the electromagnetic interactions with matter according to the tenets of relativistic quantum field theory. However Maxwell's classical equations remain mandatory for the description of a vast amount of natural phenomena. This versatility is in part due to the supplementary constitutive relations that are necessary to accommodate the wide range of materials that respond to electromagnetic fields. Although in principle such relations can be derived from the underlying quantum description of matter, in many practical situations one must rely on experimental guidance to ascertain the classical response of materials to such fields.

Once the unifying power of a spacetime formulation of physical phenomena became apparent with Einstein's relativistic world view, the natural mathematical tool for describing constitutive responses became the total stress–energy–momentum tensor for all matter and fields. Early suggestions by Minkowski [1] and Abraham [2] for the structure of its electromagnetic component in simple media initiated a long debate involving both theoretical and experimental contributions that continues to the current time (see, e.g., [3–8, 10, 25, 26]). Although it is widely recognized that this controversy is an argument about definitions [11] and that the relative merits of alternative definitions are undecidable without a complete (experimentally verifiable) covariant description of relativistic continuum mechanics for matter and fields [14], it remains important to clarify the many conflicting arguments that have appeared over the years and to offer new insights that may help in modelling the electromagnetic properties of moving media in the absence of a viable or complete description of field–particle interactions at a more fundamental level.

Some way towards this goal is offered by (covariant) averaging methods [21, 20]. These however yield non-symmetric stress–energy–momentum tensors for electromagnetic fields in simple media. If the total stress–energy–momentum is to remain symmetric this implies that other asymmetric contributions must compensate and no guidance is offered to account for such material-induced asymmetries. The need for a symmetric total stress–energy–momentum tensor is often attributed to conservation of total angular momentum despite the fact that such global conservation laws may not exist in arbitrary gravitational fields. Although the magnitude of gravitational interactions may be totally insignificant compared with the scale of those due to electromagnetism, gravity does have relevance in establishing the general framework (via the geometry of spacetime) for classical field theory and in particular this framework [15, 12] offers the most cogent means to define the total stress–energy–momentum tensor as the source of relativistic gravitation. This in turn may be related to a variational formulation [19] of the fully coupled field system of equations that underpin the classical description of interacting matter in terms of tensor (and spinor) fields on spacetime.

In this paper, stress–energy–momentum tensors are defined as variational derivatives and it is argued that the classical properties of a simple polarizable medium may be parameterized in terms of a constitutive tensor whose properties can in principle be determined by experiments in non-inertial (accelerating) frames and in the presence of weak but variable gravitational fields.

There has been a rapid development in recent years in the construction of 'traps' for confining collective states of matter on scales intermediate between macro- and micro-dimensions. Cold atoms and nano-structures offer many new avenues for technological development when coupled to probes by electromagnetic fields. The constitutive properties of such novel material will play an important role in this development. Space science is also progressing rapidly and can provide new laboratory environments with variable gravitation and controlled acceleration in which the properties of such states of matter may be explored.

It will be shown below that the response of electromagnetically polarizable media to such novel experimental environments offers a means to describe their electromagnetic constitutive properties and hence gain insight into the electromagnetic stresses induced by electromagnetic fields in such media. Supplemented with additional data based on their mechanical and elastodynamic responses one thereby gains a more confident picture of the total phenomenological stress–energy–momentum for media than that based on previous ad hoc choices.

Throughout this paper the formulation will be expressed in terms of tensor fields on spacetime with an arbitrary metric. Attention will be drawn to conservation laws when this metric admits particular symmetries. Thus, the results [16] have applicability to simple media in arbitrary gravitational fields and accommodate both media and observers with arbitrary velocities. A topical summary of the results and their relevance to current experimental findings can be found in [17].

After establishing some geometric notation, section 2 relates the electromagnetic 1-forms $\mathbf{e}, \mathbf{b}, \mathbf{d}, \mathbf{h}$ to the 2-forms F and G that enter into Maxwell's phenomenological covariant field equations in the presence of matter. Section 3 discusses stress, energy and momentum *in the vacuum* and introduces the useful notion of a *drive* form that can be used to calculate electromagnetically induced currents involving the flux of electromagnetic energy, momentum and angular momentum in Minkowski spacetime. In section 4, the definition of the stress–energy–momentum tensor is discussed with particular reference to its symmetry based on its role as a source of relativistic gravitation. The constitutive properties of the media considered in this paper are delineated in section 5 in terms of a constitutive tensor on spacetime. This includes an account of general *magneto-electric continua* and leads in section 6 to a formulation of a self-adjoint constitutive tensor describing, in general, inhomogeneous, anisotropic, magneto-electric matter in arbitrary motion. The question of an invariant characterization of magnto-electric media is mentioned in section 7. In section 8, an action principle is established to generate the phenomenological Maxwell system and the use of variational derivatives to calculate stress–energy–momentum tensors is discussed in section 9. The computation of the electromagnetic stress–energy–momentum tensor, based on the action of section 8, is non-trivial for general media exhibiting anisotropy and magneto-electric properties in arbitrary motion and is presented in some detail. The relation of this result to tensors proposed by Abraham and others is discussed in the concluding section where the relevance of the whole approach to experiments on matter in non-inertial environments with variable gravitational and electromagnetic fields is stressed.

Notations follow standard conventions with spacetime modelled as a four-dimensional, orientable, manifold M with a metric tensor field g of Lorentzian signature $(-,+,+,+)$. ΓTM denotes the set of vector fields and $\Gamma\Lambda^p M$ the set of p-form fields on M. The set $\{e^0, e^1, e^2, e^3\}$ denotes a local g-orthonormal coframe (a linearly independent collection of 1-forms) with dual frame $\{X_0, X_1, X_2, X_3\}$. If $g_{ab} = g(X_a, X_b)$, the interior contraction operator i_{X_a} with respect to X_a is written i_a with $i^a = g^{ab} i_{X_b}$, $e_b = g_{ac} e^c$ and summation over 0, 1, 2, 3. Metric duals with respect to g are written with a tilde so that $\widetilde{X} = g(X,-) \in \Gamma\Lambda^1 M$ for $X \in \Gamma TM$ and $\widetilde{\alpha} = g^{-1}(\alpha,-) \in \Gamma TM$ for $\alpha \in \Gamma\Lambda^1 M$. The Hodge dual map associated with g is denoted by $\star$. The following standard identities will be used repeatedly in subsequent sections to simplify expressions.

$$\Phi \wedge \Psi = (-1)^{pq}\Psi \wedge \Phi \qquad \text{for} \quad \Phi \in \Gamma\Lambda^p M, \quad \Psi \in \Gamma\Lambda^q M \tag{1}$$

$$\Phi \wedge \star\Psi = \Psi \wedge \star\Phi \qquad \text{for} \quad \Phi, \Psi \in \Gamma\Lambda^p M \tag{2}$$

$$i_X \star \Phi = \star(\Phi \wedge \tilde{X}) \qquad \text{for} \quad X \in \Gamma TM, \quad \Phi \in \Gamma\Lambda^p M \tag{3}$$

$$\star\, i_X \Phi = -\star \Phi \wedge \tilde{X} \qquad \text{for} \quad X \in \Gamma TM, \quad \Phi \in \Gamma\Lambda^p M \tag{4}$$

$$\star\star\Phi = (-1)^{p+1}\Phi \qquad \text{for} \quad \Phi \in \Gamma\Lambda^p M \tag{5}$$

$$i_X\Phi \wedge \Psi = (-1)^{p+1}\Phi \wedge i_X\Psi \qquad \text{for} \quad \Phi \in \Gamma\Lambda^p M, \quad \Psi \in \Gamma\Lambda^q M, \quad p+q \geqslant 5 \tag{6}$$

$$d\Phi \wedge \Psi = (-1)^{p+1}\Phi \wedge d\Psi + d(\Phi \wedge \Psi) \qquad \text{for} \quad \Phi \in \Gamma\Lambda^p M, \quad \Psi \in \Gamma\Lambda^q M \tag{7}$$

2. Electromagnetic fields

Maxwell's equations for an electromagnetic field in an arbitrary medium can be written as

$$dF = 0 \qquad \text{and} \qquad d\star G = j \tag{8}$$

where $F \in \Gamma\Lambda^2 M$ is the Maxwell 2-form, $G \in \Gamma\Lambda^2 M$ is the excitation 2-form and $j \in \Gamma\Lambda^3 M$ is the 3-form electric current source[1]. In general, the effects of gravitation and electromagnetism on matter are encoded in this system in $\star G$ and j. This dependence may be nonlinear and non-local. To close this system, 'electromagnetic constitutive relations' relating G and j to F are necessary. In the following, the medium will be considered as containing polarizable (both electrically and magnetically) matter with G restricted to a real pointwise linear function of F, thereby ignoring losses and spatial and temporal material dispersion in all frames. Continua endowed with such properties will be termed 'simple' here. The electric 4-current j will be assumed to describe (free) electric charge and plays no role in subsequent discussions.

The electric field $\mathbf{e} \in \Gamma\Lambda^1 M$ and magnetic induction field $\mathbf{b} \in \Gamma\Lambda^1 M$ associated with F are defined with respect to an arbitrary *unit* future-pointing timelike 4-velocity vector field $U \in \Gamma TM$ by

$$\mathbf{e} = i_U F \qquad \text{and} \qquad c\mathbf{b} = i_U \star F. \tag{9}$$

Since $g(U, U) = -1$

$$F = \mathbf{e} \wedge \widetilde{U} - \star(c\mathbf{b} \wedge \widetilde{U}). \tag{10}$$

The field U may be used to describe an *observer frame* on spacetime and its integral curves model idealized observers.

Likewise the displacement field $\mathbf{d} \in \Gamma\Lambda^1 M$ and the magnetic field $\mathbf{h} \in \Gamma\Lambda^1 M$ associated with G are defined with respect to U by

$$\mathbf{d} = i_U G \qquad \text{and} \qquad \mathbf{h}/c = i_U \star G. \tag{11}$$

Thus,

$$G = \mathbf{d} \wedge \widetilde{U} - \star((\mathbf{h}/c) \wedge \widetilde{U}). \tag{12}$$

It will be assumed that a material medium has associated with it a future-pointing timelike unit vector field V which may be identified with the bulk 4-velocity field of the medium in spacetime. Integral curves of V define the averaged worldlines of identifiable constituents of the medium. A *comoving observer frame with 4-velocity U* will have $U = V$.

[1] All tensors in this paper have dimensions constructed from the SI dimensions $[M], [L], [T], [Q]$ where $[Q]$ has the unit of the Coulomb in the MKS system. We adopt $[g] = [L^2], [G] = [j] = [Q], [F] = [Q]/\epsilon_0$ where the permittivity of free space ϵ_0 has the dimensions $[Q^2T^2M^{-1}L^{-3}]$ and c denotes the speed of light *in vacuo*.

3. Electromagnetic stress, energy and momentum in the vacuum

The historical development of Newtonian continuum mechanics led to the notion of a stress tensor in Euclidean 3-space that entered into the balance laws for momentum and angular momentum. With the advent of relativistic concepts this was generalized to a stress–energy–momentum tensor in spacetime giving rise to conserved quantities in situations where the metric admits symmetries.

The basic properties of the electromagnetic stress–energy–momentum tensor in the vacuum[2] can be succinctly discussed in terms of a set of 'drive' 3-forms. *In vacuo*, the Maxwell field system with a 3-form electric current source j satisfies

$$dF = 0 \qquad \text{and} \qquad \epsilon_0 d \star F = j. \tag{13}$$

For any vector field Y on spacetime and any Maxwell solution F to this system one can introduce a 'drive' 3-form associated with Y and F:

$$\tau_Y^{\mathrm{EM}} = \frac{\epsilon_0}{2c}(i_Y F \wedge \star F - i_Y \star F \wedge F). \tag{14}$$

This 3-form can be used to generate different types of conserved quantities when the vector field Y generates *(conformal) isometries* on spacetime[3]. If $\mathcal{K}$ is any (conformal) Killing vector on a domain of spacetime it then follows simply from the vacuum Maxwell-system above that

$$d\tau_{\mathcal{K}}^{\mathrm{EM}} = -\frac{1}{c} i_{\mathcal{K}} F \wedge j. \tag{15}$$

Thus for each (conformal) Killing vector field these equations describe a 'local conservation equation' ($d\tau_{\mathcal{K}} = 0$) in a source-free region ($j = 0$).

For K any *unit timelike* Killing vector one has from (14)

$$\tau_K^{\mathrm{EM}} = \frac{1}{c^2}\mathbf{e} \wedge \mathbf{h} \wedge \tilde{K} - \frac{1}{2c}\{\epsilon_0 g(\tilde{\mathbf{e}}, \tilde{\mathbf{e}}) + \mu_0 g(\tilde{\mathbf{h}}, \tilde{\mathbf{h}})\} \star \widetilde{K} \tag{16}$$

where $\mathbf{h} = \mu_0^{-1}\mathbf{b}$, $\mathbf{b}$, $\mathbf{e}$ are defined with respect to $U = K$ and $\mu_0 \equiv \frac{1}{c^2\epsilon_0}$. The spatial 2-form $\mathbf{e} \wedge \mathbf{h}$ was identified by Poynting in a source-free region as proportional to the local field energy transmitted normally across unit area per second (field energy current or power) and $\frac{1}{2}\{\epsilon_0 g(\tilde{\mathbf{e}}, \tilde{\mathbf{e}}) + \mu_0 g(\tilde{\mathbf{h}}, \tilde{\mathbf{h}})\}$ proportional to the local field energy density. More precisely, $\int_\Sigma \tau_K$ is the field energy associated with the spacelike 3-chain Σ and $\int_{S^2} i_K \tau_K$ is the power flux across an oriented spacelike 2-chain S^2.

If X is a *unit spacelike* Killing vector generating spacelike translations along open integral curves then with the split

$$\tau_X^{\mathrm{EM}} = \mu_X \wedge \tilde{V} + \mathcal{G}_X$$

where $i_V \mathcal{G}_X = 0$, the Maxwell stress 2-form μ_X may be used to identify mechanical forces produced by a flow of field momentum current or pressure with momentum density 3-form $\mathcal{G}_X$ [22].

It is important to stress that different timelike Killing vectors give rise to physically distinct notions of conserved energy. For completeness, the interpretation of 'energy' requires further information related to its mode of detection. The existence of timelike *parallel* Killing vector fields (including those whose integral curves are geodesics) are further conditions that single

[2] The notion of a classical vacuum here corresponds to spacetime devoid of all material ($j = 0$) although if j has compact support one can refer to 'vacuum domains' where $j = 0$. All regions can admit non-zero electromagnetic and gravitational fields.

[3] That means in terms of the Lie derivative $\mathcal{L}_Y$, $\mathcal{L}_Y g = \lambda g$ for some scalar λ. Y is a Killing field when $\lambda = 0$. Angular momentum currents follow in terms of Killing vector fields that generate rotational diffeomorphisms.

out particular classes that may have priority in establishing appropriate notions of conserved energy.

In general, in the absence of Killing vectors one loses strictly conserved currents (closed 3-forms) but a set of four local 3-form currents $\tau_c^{\mathrm{EM}} \equiv \tau_{X_c}^{\mathrm{EM}}$ can be defined in any local coframe. In any frame $\{X_a\}$ with dual coframe $\{e^b\}$ the 16 functions $T_{ab}^{\mathrm{EM}} = i_{X_b} \star \tau_a^{\mathrm{EM}}$ may be used to construct the tensor

$$T^{\mathrm{EM}} = T_{ab}^{\mathrm{EM}} e^a \otimes e^b \tag{17}$$

usually referred as the *stress–energy–momentum tensor* associated with the above drive forms[4].

The relationships between any stress–energy tensor T and the associated drive forms τ_a are given by

$$\tau_a = \star(T(X_a, -)) \qquad \text{and} \qquad T = \star(\tau_a \wedge e_b) e^a \otimes e^b. \tag{18}$$

In terms of the 3-forms τ_c the symmetry condition $T_{bc} = T_{cb}$ is

$$e_c \wedge \tau_b = e_b \wedge \tau_c. \tag{19}$$

4. The total stress–energy–momentum tensor

When spacetime contains domains with matter (where j may or may not be zero) such regions will in general have physical properties distinct from vacuum domains.

If a coupled system of electromagnetic, gravitational and matter fields has a *total* stress–energy–momentum tensor

$$T^{\mathrm{Total}} = T^{\mathrm{EM}} + T^{m+I} \tag{20}$$

where T^{m+I} describes matter and its interactions not included in T^{EM}, then on general grounds, if T^{Total} is symmetric, one has

$$\nabla \cdot T^{\mathrm{Total}} = 0 \tag{21}$$

in terms of a (Koszul) connection ∇ on spacetime. Different authors partition the total stress–energy tensor into a sum of partial stress–energy tensors in different ways. The divergences of certain partial stress–energy tensors are sometimes called *ponder–motive* forces.

If the connection ∇, induced from a connection on the bundle of linear frames over spacetime, is both metric-compatible and torsion-free, with gravitational fields satisfying Einstein's equations, then T^{Total} must give rise to a symmetric stress–energy tensor $T_{ab}^{\mathrm{Total}} = T_{ba}^{\mathrm{Total}}$. However any such symmetric tensor can be partitioned into non-symmetric partial tensors in infinitely many ways. Such a partition is then an expediency without fundamental significance.

In theories of gravitation based on non-pseudo-Riemannian geometries the *natural connection* may have torsion. For example, in an Einstein–Cartan theory with matter that gives rise to a connection with torsion, the generalized Einstein tensor $\mathrm{Ein}^{\mathrm{EC}}$, determined by varying the generalized Einstein–Hilbert action with respect to orthonormal coframes, is non-symmetric and hence the source tensor T^{EC} defined by

$$\mathrm{Ein}^{\mathrm{EC}} = T^{\mathrm{EC}} \tag{22}$$

is similarly non-symmetric. However for some forms of gravitational–matter couplings the variation of the total action with respect to the connection gives rise to algebraic equations

[4] In view of the above comments on the role of particular timelike and spacelike Killing vectors in constructing conserved energy–power and momentum–force currents a more coherent label for T might be the drive tensor.

for the connection[5]. In principle, these can be solved for the connection which can always be decomposed into a sum containing the torsion-free metric-compatible (Levi-Civita) connection used in Einstein's pseudo-Riemannian description of gravitation. The generalized Einstein tensor $\mathrm{Ein}^{\mathrm{EC}}$ can then be written $\mathrm{Ein} + S$ in terms of the Einstein tensor Ein and (22) becomes

$$\mathrm{Ein} = T^E \tag{23}$$

where $T^E \equiv T^{\mathrm{EC}} - S$ is symmetric and divergenceless with respect to the Levi-Civita divergence. In such cases one may define the total stress–energy tensor as the source tensor for Einstein's equations (23). It is then by definition symmetric. If the natural connection ∇ (determined by a connection variation of the total action) gives rise to dynamic torsion, determined by a partial differential system involving all fields, the reduction to a geometrical formulation in terms of a metric and Levi-Civita connection becomes an impracticality. In such a situation, the definition of the stress–energy tensor is best left as T^{EC}. This has two distinct divergences with respect to ∇ since it is not symmetric.

Such general considerations offer guidance in the construction of phenomenological partial stress–energy tensors based on either coarse-graining detailed interactions between fields or the introduction of effective degrees of freedom [13]. Indeed, such phenomenological stress–energy tensors are often of greater value than actions based on 'fundamental fields' since they can often be related more directly to experiment. Thus although in this paper gravitation will be regarded as a background interaction the electromagnetic properties of a simple medium will be accommodated into certain constitutive tensors that respond to gravitation. We then demand that an action describing such a medium in the absence of free charges gives rise by variation to Maxwell's phenomenological equations for a simple medium *and* a symmetric stress–energy tensor.

5. The constitutive tensor for simple media

In general, G may be a functional of F and properties of the medium[6].

$$G = \mathcal{Z}[F, \ldots]. \tag{24}$$

Such a functional induces, in general, nonlinear and non-local relations between **d**, **h** and **e**, **b**. These relations may be explored either empirically or by coarse graining a suitable macroscopic model. For general *linear continua* one may have for some positive integer N and collection of *constitutive tensor fields* $Z^{(r)}$ on spacetime the relation

$$G = \Sigma_{r=0}^{N} Z^{(r)}(\nabla^r F, \ldots) \tag{25}$$

in terms of some spacetime connection ∇. Additional arguments refer to variables independent of F and its derivatives. In this paper, for the simple linear media under consideration, we restrict to

$$G = Z(F)$$

for some constitutive tensor field Z. In the vacuum $G = \epsilon_0 F$.

A particularly simple linear isotropic medium may be described by a bulk 4-velocity field V, a relative permittivity scalar field ϵ and a non-vanishing relative permeability scalar field μ. In this case, Z follows from

$$\frac{G}{\epsilon_0} = \epsilon i_V F \wedge \widetilde{V} - \mu^{-1} \star (i_V \star F \wedge \widetilde{V}) \tag{26}$$

[5] For example, locally $SL(2, C)$ covariant couplings of spinor fields to gravitation fall into this category.

[6] For example, electrostriction and magnetostriction arise from the dependence of $\mathcal{Z}$ on the elastic deformation tensor of the medium [18].

$$= \left(\epsilon - \frac{1}{\mu}\right) i_V F \wedge \widetilde{V} + \frac{1}{\mu} F. \qquad (27)$$

In a comoving frame with $U = V$ (27) becomes

$$\mathbf{d} = \epsilon_0 \epsilon \mathbf{e} \qquad \text{and} \qquad \mathbf{h} = (\mu_0 \mu)^{-1} \mathbf{b}. \qquad (28)$$

For a non-magneto-electric but anisotropic medium, the relative permittivity ϵ and inverse relative permeability μ^{-1} become *spatial tensor fields* on spacetime. Thus, $\epsilon : \Gamma\Lambda^1 M \to \Gamma\Lambda^1 M$ and $\mu^{-1} : \Gamma\Lambda^1 M \to \Gamma\Lambda^1 M$ for all $\alpha \in \Gamma\Lambda^1 M$ where

$$\epsilon(\widetilde{V}) = 0, \quad i_V \epsilon(\alpha) = 0, \quad \mu^{-1}(\widetilde{V}) = 0 \quad \text{and} \quad i_V \mu^{-1}(\alpha) = 0. \qquad (29)$$

The more general constitutive relation is then given by

$$\frac{G}{\epsilon_0} = \epsilon(i_V F) \wedge \widetilde{V} - \star(\mu^{-1}(i_V \star F) \wedge \widetilde{V}) \qquad (30)$$

which in the comoving frame with $U = V$ becomes

$$\mathbf{d} = \epsilon_0 \epsilon(\mathbf{e}) \qquad \text{and} \qquad \mu_0 \mathbf{h} = \mu^{-1}(\mathbf{b}). \qquad (31)$$

Based on standard thermodynamic arguments the inverse relative permeability and relative permittivity tensors are symmetric with respect to the metric g:

$$i_{\widetilde{\alpha}} \epsilon(\beta) = i_{\widetilde{\beta}} \epsilon(\alpha) \qquad \text{and} \qquad i_{\widetilde{\alpha}} \mu^{-1}(\beta) = i_{\widetilde{\beta}} \mu^{-1}(\alpha) \qquad \text{for} \quad \alpha, \beta \in \Gamma\Lambda^1 M. \qquad (32)$$

In general, the electromagnetic fields may be related by

$$\begin{aligned} \mathbf{d} &= \zeta^{\mathrm{de}}(\mathbf{e}) + \zeta^{\mathrm{db}}(\mathbf{b}) \\ \mathbf{h} &= \zeta^{\mathrm{he}}(\mathbf{e}) + \zeta^{\mathrm{hb}}(\mathbf{b}) \end{aligned} \qquad (33)$$

where $\zeta^{\mathrm{de}}, \zeta^{\mathrm{db}}, \zeta^{\mathrm{he}}, \zeta^{\mathrm{hb}} : \Gamma\Lambda^1 M \to \Gamma\Lambda^1 M$ are spatial tensors satisfying

$$\zeta(\widetilde{V}) = i_V(\zeta(\alpha)) = 0 \qquad (34)$$

and therefore

$$\zeta(\pi_V(\alpha)) = \pi_V(\zeta(\alpha)) = \zeta(\alpha) \qquad (35)$$

for $\zeta = \zeta^{\mathrm{de}}, \zeta^{\mathrm{db}}, \zeta^{\mathrm{he}}, \zeta^{\mathrm{hb}}$ and for all $\alpha \in \Gamma\Lambda^1 M$, where π_V projects spacetime 1-forms to spatial 1-forms with respect to V, on spacetime

$$\pi_V : \Gamma\Lambda^1 M \to \Gamma\Lambda^1 M, \qquad \pi_V = \mathrm{Id}_4 + \widetilde{V} \otimes V. \qquad (36)$$

From (31), (33) it follows that if $\zeta^{\mathrm{he}} = \zeta^{\mathrm{db}} = 0$ in some frame then $\zeta^{\mathrm{de}} = \epsilon_0 \epsilon$ and $\zeta^{\mathrm{hb}} = (\mu_0 \mu)^{-1}$ in that frame. For such materials, however, one cannot assert that $\zeta^{\mathrm{he}}, \zeta^{\mathrm{db}}$ remain zero in all frames. Media with constitutive relation (33) are often referred to as *magneto-electric* [23]. We prefer to use this term to describe intrinsic magneto-electric media and will return to this point in section 7.

The tensor fields $\zeta^{\mathrm{de}}, \zeta^{\mathrm{db}}, \zeta^{\mathrm{he}}$ and ζ^{hb} are encoded into the tensor $Z : \Gamma\Lambda^2 M \to \Gamma\Lambda^2 M$ such that $G = Z(F)$. Since

$$Z(\alpha + \beta) = Z(\alpha) + Z(\beta) \qquad \text{and} \qquad Z(\lambda\alpha) = \lambda Z(\alpha) \qquad (37)$$

for all $\lambda \in \Gamma\Lambda^0 M$ and $\alpha, \beta \in \Gamma\Lambda^2 M$, the constitutive relation may be expanded in a local coframe field $\{e^0, e^1, e^2, e^3\}$ as

$$\tfrac{1}{2} G_{ab} e^a \wedge e^b = \tfrac{1}{4} Z^{cd}{}_{ab} F_{cd} e^a \wedge e^b \qquad (38)$$

where

$$Z^{cd}{}_{ab} = -Z^{cd}{}_{ba} = -Z^{dc}{}_{ab} = Z^{dc}{}_{ba}. \qquad (39)$$

These conditions alone imply that the tensor Z has 36 independent components, although additional symmetry conditions given below will reduce these to 21. From the definition of G in terms of comoving fields and (33), the relationship between Z and $\{\zeta^{\mathrm{de}}, \zeta^{\mathrm{db}}, \zeta^{\mathrm{he}}, \zeta^{\mathrm{hb}}\}$ follows as

$$Z(F) = \zeta^{\mathrm{de}}(i_V F) \wedge \widetilde{V} + \zeta^{\mathrm{db}}(i_V \star F) \wedge \widetilde{V} \\ - \star\,(\zeta^{\mathrm{he}}(i_V F) \wedge \widetilde{V}) - \star(\zeta^{\mathrm{hb}}(i_V \star F) \wedge \widetilde{V}) \tag{40}$$

and hence by contraction with V

$$\zeta^{\mathrm{de}}(\xi) = i_V Z(\xi \wedge \widetilde{V}), \qquad \zeta^{\mathrm{db}}(\xi) = -i_V Z(\star(\xi \wedge \widetilde{V})), \\ \zeta^{\mathrm{he}}(\xi) = i_V \star Z(\xi \wedge \widetilde{V}), \qquad \zeta^{\mathrm{hb}}(\xi) = -i_V \star Z(\star(\xi \wedge \widetilde{V})). \tag{41}$$

6. Symmetry of the constitutive tensor

The adjoint of any tensor $T : \Gamma\Lambda^p M \to \Gamma\Lambda^p M$ is the tensor $T^\dagger : \Gamma\Lambda^p M \to \Gamma\Lambda^p M$ defined by

$$\alpha \wedge \star T(\beta) = \beta \wedge \star T^\dagger(\alpha) \qquad \text{for} \quad \alpha, \beta \in \Gamma\Lambda^p M \tag{42}$$

Clearly $T^{\dagger\dagger} = T$. If $p = 1$, (42) gives

$$i_{\widetilde{\alpha}} T(\beta) = i_{\widetilde{\beta}} T^\dagger(\alpha) \qquad \text{for} \quad \alpha, \beta \in \Gamma\Lambda^1 M. \tag{43}$$

The symmetry conditions for the relative permittivity and inverse permeability tensors imply that ζ^{de} and ζ^{hb} are self adjoint. This symmetry is generalized to magneto-electric media:

$$\zeta^{\mathrm{de}\dagger} = \zeta^{\mathrm{de}}, \qquad \zeta^{\mathrm{hb}\dagger} = \zeta^{\mathrm{hb}} \quad \text{and} \quad \zeta^{\mathrm{db}\dagger} = -\zeta^{\mathrm{he}} \tag{44}$$

i.e. Z is assumed self-adjoint

$$Z = Z^\dagger \tag{45}$$

or raising indices with the metric

$$Z^{abcd} = Z^{cdab}. \tag{46}$$

Using sequentially (41), (4), (45), (3), (6), (41), (1), (2), (42), this condition yields

$$\alpha \wedge \star\zeta^{\mathrm{db}}(\beta) = -\alpha \wedge \star i_V Z(\star(\beta \wedge \widetilde{V})) = \alpha \wedge \widetilde{V} \wedge \star Z(\star(\beta \wedge \widetilde{V})) \\ = \star(\beta \wedge \widetilde{V}) \wedge \star Z(\alpha \wedge \widetilde{V}) = i_V \star\beta \wedge \star Z(\alpha \wedge \widetilde{V}) = \star\beta \wedge i_V \star Z(\alpha \wedge \widetilde{V}) \\ = \star\beta \wedge \zeta^{\mathrm{he}}(\alpha) = -\zeta^{\mathrm{he}}(\alpha) \wedge \star\beta = -\beta \wedge \star\zeta^{\mathrm{he}}(\alpha) = -\alpha \wedge \star\zeta^{\mathrm{he}\dagger}(\beta)$$

i.e. $\zeta^{\mathrm{db}} = -\zeta^{\mathrm{he}\dagger}$. The remaining equations in (44) follow similarly.

It follows from (39) and (46) that the number of independent components of Z reduce from 36 to 21.

7. Intrinsic magneto-electric media and the Post constraint

A constitutive tensor Z describes a *non-intrinsic-magneto-electric medium* if there exists a velocity field V for the medium such that $\zeta^{\mathrm{db}} = 0$ and $\zeta^{\mathrm{he}} = 0$. Thus, a constitutive tensor Z is intrinsically magneto-electric if there does not exist a velocity field V such that $\zeta^{\mathrm{db}} = 0$ and $\zeta^{\mathrm{he}} = 0$. If $Z(F)$ is decomposed with respect to an arbitrary frame $U \neq V$ one may find all tensors ζ non-zero, even for media that are not intrinsically magneto-electric. For a general constitutive tensor, it is a matter of linear algebra to decide whether it describes an intrinsically magneto-electric medium or not.

A useful characterization of magneto-electric media may be given in terms of invariants constructed from Z and the metric. One such invariant introduced by Post [9, 24] is

$$\chi = i_a i_b \star (Z(e^a \wedge e^b)). \tag{47}$$

In terms of spatial tensors with respect the medium velocity V

$$\begin{aligned}
\chi &= i_a i_b \star (Z(e^a \wedge e^b)) \\
&= i_a i_b \star (\zeta^{\mathrm{de}}(i_V(e^a \wedge e^b)) \wedge \widetilde{V}) + i_a i_b \star (\zeta^{\mathrm{db}}(i_V \star (e^a \wedge e^b)) \wedge \widetilde{V}) \\
&\quad + i_a i_b (\zeta^{\mathrm{he}}(i_V(e^a \wedge e^b)) \wedge \widetilde{V}) + i_a i_b (\zeta^{\mathrm{hb}}(i_V \star (e^a \wedge e^b)) \wedge \widetilde{V}) \\
&= i_a i_b \star (\zeta^{\mathrm{db}}(i_V \star (e^a \wedge e^b)) \wedge \widetilde{V}) + i_a i_b (\zeta^{\mathrm{he}}(i_V(e^a \wedge e^b)) \wedge \widetilde{V})
\end{aligned} \tag{48}$$

since

$$i_a i_b \star (\zeta^{\mathrm{de}}(i_V(e^a \wedge e^b)) \wedge \widetilde{V}) = i_a i_b (\zeta^{\mathrm{hb}}(i_V \star (e^a \wedge e^b)) \wedge \widetilde{V}) = 0.$$

Using

$$\star(e^a \wedge e^b \wedge \widetilde{V}) = \star(e^a \wedge e^b \wedge e^c) V_c = V_c \star (e^a \wedge e^b \wedge e^c \wedge e^d) e_d = -\varepsilon^{abcd} V_c e_d$$

and

$$\star(\xi \wedge \widetilde{V} \wedge e_b \wedge e_a) = \star(e_a \wedge e_b \wedge \widetilde{V} \wedge \xi) = \star(e_a \wedge e_b \wedge e_e \wedge e_f) V^e \xi^f = -\varepsilon_{abef} V^e \xi^f$$

with $\varepsilon_{abef}\varepsilon^{abcd} = \delta^d_e \delta^c_f - \delta^c_e \delta^d_f$, the first term on the last line of (48) yields

$$\begin{aligned}
i_a i_s \star (\zeta^{\mathrm{db}}(i_V \star (e^a \wedge e^s)) \wedge \widetilde{V}) &= \star(\zeta^{\mathrm{db}}(\star(e^a \wedge e^s \wedge \widetilde{V})) \wedge \widetilde{V} \wedge e_s \wedge e_a) \\
&= -\star(\zeta^{\mathrm{db}}(e_r) \wedge \widetilde{V} \wedge e_s \wedge e_a)\varepsilon^{ascr} V_c \\
&= i^f \zeta^{\mathrm{db}}(e_r)\varepsilon_{asef}\varepsilon^{ascr} V^e V_c = 2 i_a \zeta^{\mathrm{db}}(e^a)
\end{aligned}$$

while the second term is

$$\begin{aligned}
i_a i_b (\zeta^{\mathrm{he}}(i_V(e^a \wedge e^b)) \wedge \widetilde{V}) &= i_a i_b (\zeta^{\mathrm{he}}(v^a e^b) \wedge \widetilde{V}) - i_a i_b (\zeta^{\mathrm{he}}(v^b e^a) \wedge \widetilde{V}) \\
&= 2 i_V i_b (\zeta^{\mathrm{he}}(e^b) \wedge \widetilde{V}) = -2 i_b \zeta^{\mathrm{he}}(e^b).
\end{aligned}$$

Hence, using (44)

$$\chi = 4 i_a \zeta^{\mathrm{db}}(e^a) = -4 i_a \zeta^{\mathrm{he}}(e^a). \tag{49}$$

Thus since V is an arbitrary medium velocity, a sufficient condition for a medium to be intrinsically magneto-electric is that $\chi \neq 0$.

However some intrinsically magneto-electric media may have $\chi = 0$. For example, consider the self-adjoint constitutive tensor given, in some local orthonormal coframe $\{e^0, e^1, e^2, e^3\}$, by

$$Z(F) = F_{23} e^0 \wedge e^1 + F_{13} e^0 \wedge e^2 - F_{02} e^1 \wedge e^3 - F_{01} e^2 \wedge e^3.$$

Then with $V = X_0$

$$\zeta^{\mathrm{db}}(\xi) = (i_1 \xi) e^1 - (i_2 \xi) e^2 \qquad \text{and} \qquad \zeta^{\mathrm{he}}(\xi) = -(i_1 \xi) e^1 + (i_2 \xi) e^2$$

and $\chi = 0$. However one easily verifies that $\zeta^{\mathrm{db}} \neq 0$ with respect to any arbitrary unit timelike V. Hence, Z describes an intrinsically magneto-electric medium.

A minimal set of invariants whose non-vanishing is a necessary and sufficient condition for a medium to be intrinsically magneto-electric is not known to the authors.

8. Action for source-free electromagnetic fields in a simple medium

The classical equations describing the total system of matter and fields will be considered as arising from the extremum of some *total action functional* under suitable variations with compact support. This action should be constructed from an action density 4-form on spacetime in terms of (pull-backs) of sections (and their derivatives) of field bundles carrying representations of local symmetry (gauge) groups and maps between them. Observed local symmetries in nature arise in such a formalism by ensuring that the action 4-form is a scalar under local changes of section. To maintain these covariances appropriate connections are required to define tensorial (and spinorial) covariant derivatives of sections. In addition, the action may depend on tensor-valued functions of these sections. All variational principles require a specification of what objects in the action are to be varied and these then constitute the dynamical variables of the theory. In the following, we concentrate on a contribution Λ to the total action arising from the effects of the electromagnetic field and gravitation in different types of 'media'. We exclude from this Λ the interaction with charged matter and the dynamics of the gravitational field itself. Included is the effect of the electromagnetic field on a polarizable and magnetizable medium assumed to be described in terms of a particular constitutive tensor Z. In particular, we explore how the response of the medium to gravitation as well as the electromagnetic field can be used to establish the stress–energy–momentum tensor associated with different choices of constitutive tensor. Thus, the action 4-form Λ will be taken to depend only on the spacetime metric and the class of Maxwell 1-form potentials A with $F = dA$. The dependence of the tensor field Z on these variables will be explored in some detail below.

We have insisted that in the absence of free charge the electromagnetic fields F and G for a simple medium in any spacetime metric must satisfy

$$dF = 0 \qquad \text{and} \qquad d \star G = 0 \tag{50}$$

Before generating an electromagnetic stress–energy–momentum tensor from a particular contribution to the total action it is necessary to verify that these field equations arise by suitable variation. Consider then the contribution $S[A, g] = \int_M \Lambda$ where $F = dA$, $G = Z(F)$ with $Z = Z^\dagger$ and

$$c\Lambda = \tfrac{1}{2} F \wedge \star G = \tfrac{1}{2} F \wedge \star Z(F). \tag{51}$$

If a prime denotes the variation with respect to A, then working modulo d:

$$\begin{aligned} c\Lambda' &= \tfrac{1}{2}(dA' \wedge \star Z(dA) + dA \wedge \star Z(dA')) \\ &= dA' \wedge \star Z(dA) = A' \wedge d \star Z(dA) = A' \wedge d \star G. \end{aligned}$$

Hence, the source-free Maxwell equations (50) follow by variation with respect to A from the action (51). Note that the symmetry condition (45) of the tensor Z is essential in this variation.

9. Variational derivatives and stress–energy–momentum tensors

To effect the metric variations of the above action functional let $t \to g_t$ be a curve in the space of Lorentzian signatured metrics, with $g_0 \equiv g_t|_{t=0}$. The 'tangent' to the curve $t \to g_t$ at the point $t = 0$ is written as $\dot{g}$:

$$g_t = g_0 + t\dot{g} + O(t^2). \tag{52}$$

For a general object K which may be a tensor or a map which depends on the metric g, write similarly $t \to K_t$ as the one-parameter set of objects encoding the dependence of K on

g_t, $K_0 = K_t|_{t=0}$ and $\dot{K} = \frac{\mathrm{d}}{\mathrm{d}t} K_t\big|_{t=0}$, so

$$K_t = K_0 + t\dot{K} + O(t^2), \tag{53}$$

K_t will be referred to as the metric induced *lift* of K.

One may represent the local variation g_t in different ways. One way is to vary the components of g_t with respect to a fixed local coframe $\{e_0^a\}$, i.e.

$$g_t = (g_t)_{ab} e_0^a \otimes e_0^b \qquad \text{where} \quad (g_t)_{ab} = g_t((X_0)_a, (X_0)_b). \tag{54}$$

One can set the fixed frame to be orthonormal with respect to the unvaried metric so that $(g_0)_{ab} = \eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$. The derivative $\dot{g}$ is therefore given by

$$\dot{g} = \dot{g}_{ab} e_0^a \otimes e_0^b. \tag{55}$$

Alternatively, one may vary the coframe, i.e. choose a one-parameter set of coframes $t \to e_t^a$ for $a = 0, \ldots, 3$ such that $e_t^a\big|_{t=0} = e_0^a$ and

$$g_t = \eta_{ab} e_t^a \otimes e_t^b. \tag{56}$$

The derivative of $t \to e_t^a$ at $t = 0$ follows from

$$e_t^a = e_0^a + t\dot{e}^a + O(t^2). \tag{57}$$

The derivative $\dot{g}$ may also therefore be written as

$$\dot{g} = \eta_{ab}\left(\dot{e}^a \otimes e_0^b + e_0^a \otimes \dot{e}^b\right). \tag{58}$$

The drive 3-forms τ_a associated with any action 4-form Λ are defined by the variation of Λ with respect to the orthonormal coframe as

$$\dot{\Lambda} = \dot{e}^a \wedge \tau_a. \tag{59}$$

If the variation of Λ with respect to the orthonormal coframe is induced entirely from the metric g (and the metric-compatible torsion-free Levi-Civita connection) then

$$\tau_a = 2 i_{X_b}\left(\frac{\delta\Lambda}{\delta g_{bc}}\right)\eta_{ac}. \tag{60}$$

This follows immediately by equating (55) and (58):

$$\dot{g}_{ab} e_0^a \otimes e_0^b = \eta_{ab}\left(\dot{e}^a \otimes e_0^b + e_0^a \otimes \dot{e}^b\right)$$

so

$$\dot{g}_{ab} = \eta_{cd}\left(\dot{e}^c(X_a)\delta_b^d + \delta_a^c \dot{e}^d(X_b)\right) = \dot{e}_a(X_b) + \dot{e}_b(X_a) = i_b\dot{e}_a + i_a\dot{e}_b$$

since one may drop the 0 subscript here without ambiguity: $X_a = (X_0)_a$. Then,

$$\dot{\Lambda} = \dot{e}^a \wedge \tau_a = \frac{\delta\Lambda}{\delta g_{ab}}\dot{g}_{ab} = \frac{\delta\Lambda}{\delta g_{ab}}(\dot{e}_a(X_b) + \dot{e}_b(X_a)) = 2\frac{\delta\Lambda}{\delta g_{ab}}\dot{e}_a(X_b)$$
$$= 2\dot{e}_a \wedge i_b\left(\frac{\delta\Lambda}{\delta g_{ab}}\right).$$

By (18) the tensor associated to τ_a is given by

$$T = -2\star\left(\frac{\delta\Lambda}{\delta g_{ab}}\right) e_a \otimes e_b \tag{61}$$

and is manifestly symmetric.

In the following, it is necessary to make explicit the metric dependence of various elements that enter in the action 4-form Λ and in particular to pass between vector fields and forms using the varied metric g_t. Thus, the notations $g_t : \Gamma TM \to \Gamma\Lambda^1 M, X \mapsto g_t(X)$ and

$g_t^{-1} : \Gamma\Lambda^1 M \to \Gamma TM, \alpha \mapsto g_t^{-1}(\alpha)$ for the metric dual of vectors and 1-forms with respect to the metric g_t are used. For vectors or 1-forms which already have a subscript 0 or t we continue to use the tilde notation without ambiguity so that, for example, the g_t metric dual of the vector X_t can be written as $\widetilde{X}_t \equiv g_t(X_t)$.

Following (53) one has the maps $\star_0$, $\star_t$ and $\dot\star$ and from the Leibnitz rule (evaluated at $t = 0$)

$$(\star\alpha)^{\cdot} = \dot\star\alpha + \star\dot\alpha \tag{62}$$

for all $\alpha_t \in \Gamma\Lambda^p M$. It follows simply (see the appendix) that

$$\dot\star\alpha = \dot e^a \wedge i_a \star \alpha - \star(\dot e^a \wedge i_a\alpha) \qquad \text{for} \quad \alpha_t \in \Gamma\Lambda^p M. \tag{63}$$

Taking the derivative of $\Phi \wedge \star_t \Psi = \Psi \wedge \star_t \Phi$ with respect to t gives

$$\Phi \wedge \dot\star\Psi = \Psi \wedge \dot\star\Phi \qquad \text{for} \quad \Phi, \Psi \in \Gamma\Lambda^p M. \tag{64}$$

Thus with the metric induced lift of the constitutive tensor Z

$$Z_t = Z_0 + t\dot Z + O(t^2) \tag{65}$$

one writes

$$(\star Z_t(F))^{\cdot} = \dot\star Z(F) + \star\dot Z(F). \tag{66}$$

Since there is a one-parameter set of Hodge duals, we need to distinguish $\dagger_t$ and $\dagger_0$. Furthermore, (42) becomes

$$\alpha \wedge \star_t T(\beta) = \beta \wedge \star_t T^{\dagger_t}(\alpha) \tag{67}$$

for all $\alpha, \beta \in \Gamma\Lambda^p M$ and (43) becomes

$$i_{g_t^{-1}\alpha} T(\beta) = i_{g_t^{-1}\beta} T^{\dagger_t}(\alpha) \tag{68}$$

for all $\alpha, \beta \in \Gamma\Lambda^1 M$.

10. Computation of the stress–energy–momentum tensor

In this section, the variation of the above action (51) is explored for a particular choice of the metric dependence for Z_t, corresponding to a perturbative response of the medium to gravitation.

For a general lift, the action 4-form (51) is written as

$$c\Lambda_t = \tfrac{1}{2} F \wedge \star_t Z_t(F) \tag{69}$$

hence

$$c\dot\Lambda = \tfrac{1}{2}(F \wedge \dot\star Z(F) + F \wedge \star\dot Z(F)). \tag{70}$$

From (63)

$$\begin{aligned} F \wedge \dot\star Z(F) &= F \wedge \dot\star G = F \wedge \dot e^a \wedge i_a \star G - F \wedge \star(\dot e^a \wedge i_a G) \\ &= \dot e^a \wedge F \wedge i_a \star G - \dot e^a \wedge i_a G \wedge \star F \\ &= \dot e^a \wedge (F \wedge i_a \star G - i_a G \wedge \star F) \end{aligned} \tag{71}$$

and

$$F \wedge \star\dot Z(F) = 2F \wedge \star\frac{\delta Z}{\delta g_{ab}}(F) i_b \dot e^a = 2\dot e^a \wedge i_b \left(F \wedge \star\frac{\delta Z}{\delta g_{ab}}(F)\right).$$

Therefore, the drive forms are given by

$$c\tau_a = \frac{1}{2} F \wedge i_a \star G - \frac{1}{2} i_a G \wedge \star F + i_b \left(F \wedge \star\frac{\delta Z}{\delta g_{ab}}(F)\right). \tag{72}$$

For a physical medium with bulk motion that can sustain elastic stresses associated with its atomic constituents one expects that the history of such bulk motion should have some influence on the constitutive properties via some associated 4-velocity field[7]. To include the possible dependence of the stress–energy–momentum tensor on such bulk motion of the medium one requires Z to depend on this motion in some manner. In (33) Z is specified in terms of electromagnetic fields measured in the comoving frame V of the medium. It is therefore natural to prescribe a lift of this expression involving the lifts of V_0 and $\{\zeta_0^{\text{de}}, \zeta_0^{\text{db}}, \zeta_0^{\text{he}}, \zeta_0^{\text{hb}}\}$. The natural lift of the medium velocity V_0 is

$$V_t = \frac{V_0}{\sqrt{-g_t(V_0, V_0)}}. \tag{73}$$

The metric dual of V_t is given by $\tilde{V}_t = g_t(V_t)$ and the projection π_V (36) is lifted to

$$\pi_t = \text{Id}_4 + \tilde{V}_t \otimes V_t. \tag{74}$$

The decomposition (41) of Z_0 and Z_t with respect to the medium velocities V_0 and V_t is given by $\{\zeta_0^{\text{de}}, \zeta_0^{\text{db}}, \zeta_0^{\text{he}}, \zeta_0^{\text{hb}}\}$ and $\{\zeta_t^{\text{de}}, \zeta_t^{\text{db}}, \zeta_t^{\text{he}}, \zeta_t^{\text{hb}}\}$, respectively, following the notation (53).

The lifted tensors $\zeta_t^{\text{de}}, \zeta_t^{\text{db}}, \zeta_t^{\text{he}}$ and ζ_t^{hb} will be now chosen to satisfy three properties:

- For all t in the neighbourhood of $t = 0$

$$\zeta_t|_{t=0} = \zeta_0 \qquad \text{for} \quad \zeta_t = \zeta_t^{\text{de}}, \zeta_t^{\text{db}}, \zeta_t^{\text{he}}, \zeta_t^{\text{hb}}. \tag{75}$$

- For all t in the neighbourhood of $t = 0$ they map the vector space that is g_t-orthogonal to $\tilde{V}_t$ to itself. This is achieved by lifting (34):

$$\zeta_t(\tilde{V}_t) = 0 \qquad \text{and} \qquad i_{V_t}\zeta_t(\alpha) = 0 \tag{76}$$

  for $\zeta_t = \zeta_t^{\text{de}}, \zeta_t^{\text{db}}, \zeta_t^{\text{he}}, \zeta_t^{\text{hb}}$ and all $\alpha \in \Gamma\Lambda^1 M$. The corresponding lift of (35) is

$$\zeta_t(\pi_t(\alpha)) = \pi_t(\zeta_t(\alpha)) = \zeta_t(\alpha). \tag{77}$$

- For all t in the neighbourhood of $t = 0$ they retain the adjoint conditions (44):

$$\left(\zeta_t^{\text{de}}\right)^{\dagger_t} = \zeta_t^{\text{de}}, \qquad \left(\zeta_t^{\text{hb}}\right)^{\dagger_t} = \zeta_t^{\text{hb}} \qquad \text{and} \qquad \left(\zeta_t^{\text{db}}\right)^{\dagger_t} = -\zeta_t^{\text{he}}. \tag{78}$$

These requirements are all satisfied by setting

$$i_X\zeta_t(\alpha) = \tfrac{1}{2}\left(i_X\zeta_0(\pi_t\alpha) + i_{g_t^{-1}\alpha}(\zeta_0)^{\dagger_0}(\pi_t g_t X)\right) \tag{79}$$

for all $\alpha \in \Gamma\Lambda^1 M$ and $X \in \Gamma TM$, i.e.[8]

$$\begin{aligned}
i_X\zeta_t^{\text{de}}(\alpha) &= \tfrac{1}{2}\left(i_X\zeta_0^{\text{de}}(\pi_t\alpha) + i_{g_t^{-1}\alpha}\zeta_0^{\text{de}}(\pi_t g_t X)\right),\\
si_X\zeta_t^{\text{hb}}(\alpha) &= \tfrac{1}{2}\left(i_X\zeta_0^{\text{hb}}(\pi_t\alpha) + i_{g_t^{-1}\alpha}\zeta_0^{\text{hb}}(\pi_t g_t X)\right),\\
i_X\zeta_t^{\text{db}}(\alpha) &= \tfrac{1}{2}\left(i_X\zeta_0^{\text{db}}(\pi_t\alpha) - i_{g_t^{-1}\alpha}\zeta_0^{\text{he}}(\pi_t g_t X)\right),\\
i_X\zeta_t^{\text{he}}(\alpha) &= \tfrac{1}{2}\left(i_X\zeta_0^{\text{he}}(\pi_t\alpha) - i_{g_t^{-1}\alpha}\zeta_0^{\text{db}}(\pi_t g_t X)\right).
\end{aligned} \tag{80}$$

To verify that (79) obeys (75) note that from (36)

$$g_0^{-1}\pi_0 g_0 X = X + g_0(X, V_0)V_0$$

[7] Relativistic strings and membranes with dynamics that arise from re-parameterization independent actions are an exception since, without 'constituents', no preferred parametrization of their histories should be identified.

[8] For an isotropic, non-magneto-electric medium (27) and (28), the lifts (80) reduces to the lifts

$$\zeta_t^{\text{de}} = \epsilon\pi_t, \qquad \zeta_t^{\text{hb}} = \mu^{-1}\pi_t, \qquad \zeta_t^{\text{db}} = 0 \qquad \text{and} \qquad \zeta_t^{\text{he}} = 0$$

which in a comoving frame yield the relations

$$\mathbf{d}_t = \epsilon_0\epsilon\mathbf{e}_t \qquad \text{and} \qquad \mathbf{h}_t = (\mu_0\mu)^{-1}\mathbf{b}_t$$

where the scalars ϵ and μ^{-1} are independent of the ambient metric.

and hence from (68) and (34)

$$i_{g_0^{-1}\alpha}(\zeta_0)^{\dagger_0}(\pi_0 g_0 X) = i_{g_0^{-1}\pi_0 g_0 X}\zeta_0(\alpha) = i_X\zeta_0(\alpha).$$

Thus at $t = 0$ (79) becomes

$$i_X\zeta_t(\alpha)|_{t=0} = \tfrac{1}{2}\big(i_X\zeta_0(\pi_0\alpha) + i_{g_0^{-1}\alpha}(\zeta_0)^{\dagger_0}(\pi_0 g_0 X)\big) = i_X\zeta_0(\alpha)$$

using (35).

To verify that (79) obeys (76) observe that

$$i_X\zeta_t(\tilde{V}_t) = \tfrac{1}{2}\big(i_X\zeta_0(\pi_t\tilde{V}_t) + i_{g_t^{-1}\tilde{V}_t}(\zeta_0)^{\dagger_0}(\pi_t g_t X)\big).$$

Now $\pi_t\tilde{V}_t = 0$ so the first term vanishes. Also using (68)

$$\begin{aligned} i_{g_t^{-1}\tilde{V}_t}(\zeta_0)^{\dagger_0}(\pi_t g_t X) &= i_{V_t}(\zeta_0)^{\dagger_0}(\pi_t g_t X) = \sqrt{-g_t(V_0, V_0)}\, i_{V_0}(\zeta_0)^{\dagger_0}(\pi_t g_t X) \\ &= \sqrt{-g_t(V_0, V_0)}\, i_{g_t^{-1}\pi_t g_t X}\zeta_0(\tilde{V}_0) = 0. \end{aligned}$$

Hence $\zeta_t(\tilde{V}_t) = 0$. Likewise

$$i_{V_t}\zeta_t(\alpha) = \tfrac{1}{2}\big(i_{V_t}\zeta_0(\pi_t\alpha) + i_{g_t^{-1}\alpha}(\zeta_0)^{\dagger_0}(\pi_t\tilde{V}_t)\big) = 0.$$

Finally to verify that (80) obeys (78) use (68) and (80) twice:

$$\begin{aligned} i_{g_t^{-1}\alpha}\big(\zeta_t^{\mathrm{db}}\big)^{\dagger_t}(\beta) &= i_{g_t^{-1}\beta}\zeta_t^{\mathrm{db}}(\alpha) = \tfrac{1}{2}\big(i_{g_t^{-1}\beta}\zeta_0^{\mathrm{db}}(\pi_t\alpha) - i_{g_t^{-1}\alpha}\zeta_0^{\mathrm{he}}(\pi_t\beta)\big) \\ &= -i_{g_t^{-1}\alpha}\zeta_t^{\mathrm{he}}(\beta). \end{aligned}$$

In a similar way it follows that $\big(\zeta_t^{\mathrm{de}}\big)^{\dagger_t} = \zeta_t^{\mathrm{de}}$ and $\big(\zeta_t^{\mathrm{hb}}\big)^{\dagger_t} = \zeta_t^{\mathrm{hb}}$. Thus (80) provide natural conditions for the lifts (75) to (78)[9].

Inserting the relations (80) into $Z(F)$, (40), the action 4-form (69) becomes

$$\begin{aligned} 2c\Lambda_t = {} & F \wedge \star_t\big(\zeta_t^{\mathrm{de}}\big(i_{V_t}F\big) \wedge \tilde{V}_t\big) + F \wedge \star_t\big(\zeta_t^{\mathrm{db}}\big(i_{V_t}\star_t F\big) \wedge \tilde{V}_t\big) \\ & + F \wedge \zeta_t^{\mathrm{he}}\big(i_{V_t}F\big) \wedge \tilde{V}_t + F \wedge \zeta_t^{\mathrm{hb}}\big(i_{V_t}\star_t F\big) \wedge \tilde{V}_t. \end{aligned} \tag{81}$$

To ease the density of notation in the following, the symbol $\mathbf{b}_t$ now stands for $c\mathbf{b}_t$ and $\mathbf{h}_t$ stands for $\frac{\mathbf{h}_t}{c}$. The lifts

$$\mathbf{e}_t = i_{V_t}F, \qquad \mathbf{b}_t = i_{V_t}\star_t F, \qquad \widetilde{\mathbf{e}}_t = g_t^{-1}\big(i_{V_t}F\big) \qquad \text{and} \qquad \widetilde{\mathbf{b}}_t = g_t^{-1}\big(i_{V_t}\star_t F\big) \tag{82}$$

satisfy

$$\pi_t\mathbf{e}_t = \mathbf{e}_t \qquad \text{and} \qquad \pi_t\mathbf{b}_t = \mathbf{b}_t. \tag{83}$$

Sequentially using (3), (6), (4), (80), (83), (4), (6), (3), the first term on the right-hand side of (81) becomes

$$\begin{aligned} F \wedge \star_t(\zeta_t^{\mathrm{de}}(\mathbf{e}_t) \wedge \tilde{V}_t) &= F \wedge i_{V_t}\star_t\zeta_t^{\mathrm{de}}(\mathbf{e}_t) = -\mathbf{e}_t \wedge \star_t\zeta_t^{\mathrm{de}}(\mathbf{e}_t) = -(\star_t 1) i_{\widetilde{\mathbf{e}}_t}\zeta_t^{\mathrm{de}}(\mathbf{e}_t) \\ &= -\tfrac{1}{2}(\star_t 1)\left(i_{\widetilde{\mathbf{e}}_t}\zeta_0^{\mathrm{de}}(\pi_t\mathbf{e}_t) + i_{\widetilde{\mathbf{e}}_t}\zeta_0^{\mathrm{de}}(\pi_t\mathbf{e}_t)\right) = -(\star_t 1) i_{\widetilde{\mathbf{e}}_t}\zeta_0^{\mathrm{de}}(\mathbf{e}_t) \\ &= -\mathbf{e}_t \wedge \star_t\zeta_0^{\mathrm{de}}(\mathbf{e}_t) = F \wedge i_{V_t}\star_t\zeta_0^{\mathrm{de}}(\mathbf{e}_t) = F \wedge \star_t(\zeta_0^{\mathrm{de}}(\mathbf{e}_t) \wedge \tilde{V}_t). \end{aligned}$$

[9] The requirements (75)–(78) are not meant to be exhaustive. Other lifts ζ_t could involve gradients of the spacetime metric corresponding to gravitational tidal effects on the constitutive tensor. For example if $\mathcal{R}$ is the curvature scalar associated with the Levi-Civita connection then the lifts

$$i_X\zeta_t(\alpha) = \tfrac{1}{2}(\mathcal{R}_t - \mathcal{R}_0 + 1)(i_X\zeta_0(\pi_t\alpha) + i_{g_t^{-1}\alpha}(\zeta_0)^{\dagger_0}(\pi_t g_t X))$$

also satisfy (75)–(78).

Similarly sequentially using (3), (6), (4), (80), (83), (4), (6), (3), (2), (5), the second term on the right-hand side of (81) gives

$$\begin{aligned}
&F \wedge \star_t\big(\zeta_t^{\rm db}(\mathbf{b}_t) \wedge \tilde{V}_t\big)\\
&\quad = F \wedge i_{V_t}\star_t\zeta_t^{\rm db}(\mathbf{b}_t) = -\mathbf{e}_t \wedge \star_t\zeta_t^{\rm db}(\mathbf{b}_t) = -(\star_t 1)i_{\widetilde{\mathbf{e}}_t}\zeta_t^{\rm db}(\mathbf{b}_t)\\
&\quad = -\tfrac{1}{2}(\star_t 1)\left(i_{\widetilde{\mathbf{e}}_t}\zeta_0^{\rm db}(\pi_t\mathbf{b}_t) - i_{\widetilde{\mathbf{b}}_t}\zeta_0^{\rm he}(\pi_t\mathbf{e}_t)\right) = -\tfrac{1}{2}(\star_t 1)\left(i_{\widetilde{\mathbf{e}}_t}\zeta_0^{\rm db}(\mathbf{b}_t) - i_{\widetilde{\mathbf{b}}_t}\zeta_0^{\rm he}(\mathbf{e}_t)\right)\\
&\quad = -\tfrac{1}{2}\mathbf{e}_t \wedge \star_t\zeta_0^{\rm db}(\mathbf{b}_t) + \tfrac{1}{2}\mathbf{b}_t \wedge \star_t\zeta_0^{\rm he}(\mathbf{e}_t) = \tfrac{1}{2}F \wedge i_{V_t}\star_t\zeta_0^{\rm db}(\mathbf{b}_t) - \tfrac{1}{2}\star_t F \wedge i_{V_t}\star_t\zeta_0^{\rm he}(\mathbf{e}_t)\\
&\quad = \tfrac{1}{2}F \wedge \star_t\big(\zeta_0^{\rm db}(\mathbf{b}_t) \wedge \tilde{V}_t\big) - \tfrac{1}{2}\star_t\big(\zeta_0^{\rm he}(\mathbf{e}_t) \wedge \tilde{V}_t\big) \wedge \star_t F\\
&\quad = \tfrac{1}{2}F \wedge \star_t\big(\zeta_0^{\rm db}(\mathbf{b}_t) \wedge \tilde{V}_t\big) - \tfrac{1}{2}F \wedge \star_t\star_t\big(\zeta_0^{\rm he}(\mathbf{e}_t) \wedge \tilde{V}_t\big)\\
&\quad = \tfrac{1}{2}F \wedge \star_t\big(\zeta_0^{\rm db}(\mathbf{b}_t) \wedge \tilde{V}_t\big) + \tfrac{1}{2}F \wedge \zeta_0^{\rm he}(\mathbf{e}_t) \wedge \tilde{V}_t.
\end{aligned}$$

It is useful to record from this calculation that

$$-\tfrac{1}{2}(\star_t 1)\big(i_{\widetilde{\mathbf{e}}_t}\zeta_0^{\rm db}(\mathbf{b}_t) - i_{\widetilde{\mathbf{b}}_t}\zeta_0^{\rm he}(\mathbf{e}_t)\big) = \tfrac{1}{2}F \wedge \star_t\big(\zeta_0^{\rm db}(\mathbf{b}_t) \wedge \tilde{V}_t\big) + \tfrac{1}{2}F \wedge \zeta_0^{\rm he}(\mathbf{e}_t) \wedge \tilde{V}_t. \qquad (84)$$

Sequentially using (5), (4), (2), (4), (80), (83), (84), the third term on the right-hand side of (81) yields

$$\begin{aligned}
F \wedge \zeta_t^{\rm he}(\mathbf{e}_t) \wedge \tilde{V}_t &= -\zeta_t^{\rm he}(\mathbf{e}_t) \wedge \tilde{V}_t \wedge \star_t\star_t F = \zeta_t^{\rm he}(\mathbf{e}_t) \wedge \star_t\mathbf{b}_t = \mathbf{b}_t \wedge \star_t\zeta_t^{\rm he}(\mathbf{e}_t)\\
&= (\star_t 1)i_{\widetilde{\mathbf{b}}_t}\zeta_t^{\rm he}(\mathbf{e}_t) = \tfrac{1}{2}(\star_t 1)\left(i_{\widetilde{\mathbf{b}}_t}\zeta_0^{\rm he}(\pi_t\mathbf{e}_t) - i_{\widetilde{\mathbf{e}}_t}\zeta_0^{\rm db}(\pi_t\mathbf{b}_t)\right)\\
&= \tfrac{1}{2}(\star_t 1)\left(i_{\widetilde{\mathbf{b}}_t}\zeta_0^{\rm he}(\mathbf{e}_t) - i_{\widetilde{\mathbf{e}}_t}\zeta_0^{\rm db}(\mathbf{b}_t)\right)\\
&= \tfrac{1}{2}F \wedge \star_t\big(\zeta_0^{\rm db}(\mathbf{b}_t) \wedge \tilde{V}_t\big) + \tfrac{1}{2}F \wedge \zeta_0^{\rm he}(\mathbf{e}_t) \wedge \tilde{V}_t.
\end{aligned}$$

Finally, on sequential use of (5), (4), (2), (4), (80), (83)

$$\begin{aligned}
F \wedge \zeta_t^{\rm hb}(\mathbf{b}_t) \wedge \tilde{V}_t &= -\zeta_t^{\rm hb}(\mathbf{b}_t) \wedge \tilde{V}_t \wedge \star_t\star_t F = \zeta_t^{\rm hb}(\mathbf{b}_t) \wedge \star_t\mathbf{b}_t = \mathbf{b}_t \wedge \star_t\zeta_t^{\rm hb}(\mathbf{b}_t)\\
&= (\star_t 1)i_{\widetilde{\mathbf{b}}_t}\zeta_t^{\rm hb}(\mathbf{b}_t) = \tfrac{1}{2}(\star_t 1)\left(i_{\widetilde{\mathbf{b}}_t}\zeta_0^{\rm hb}(\pi_t\mathbf{b}_t) + i_{\widetilde{\mathbf{b}}_t}\zeta_0^{\rm hb}(\pi_t\mathbf{b}_t)\right)\\
&= (\star_t 1)i_{\widetilde{\mathbf{b}}_t}\zeta_0^{\rm hb}(\mathbf{b}_t)
\end{aligned}$$

and so by reversing this sequence of steps

$$F \wedge \zeta_t^{\rm hb}(\mathbf{b}_t) \wedge \tilde{V}_t = F \wedge \zeta_0^{\rm hb}(\mathbf{b}_t) \wedge \tilde{V}_t.$$

Hence (81) simplifies to

$$\begin{aligned}
2c\Lambda_t &= F \wedge \star_t\big(\zeta_0^{\rm de}\big(i_{V_t}F\big) \wedge \tilde{V}_t\big) + F \wedge \star_t\big(\zeta_0^{\rm db}\big(i_{V_t}\star_t F\big) \wedge \tilde{V}_t\big)\\
&\quad + F \wedge \zeta_0^{\rm he}\big(i_{V_t}F\big) \wedge \tilde{V}_t + F \wedge \zeta_0^{\rm hb}\big(i_{V_t}\star_t F\big) \wedge \tilde{V}_t,
\end{aligned} \qquad (85)$$

i.e., the constitutive tensors ζ_t in the action may be replaced by ζ_0 and hence the metric dependence of Λ_t is seen to reside solely in $\star_t$, V_t and $\tilde{V}_t$.

The derivative of (85) at $t = 0$ is given by

$$\begin{aligned}
c\dot{\Lambda} &= F \wedge \dot{\star}(\zeta^{\rm de}(i_V F) \wedge \widetilde{V}) + F \wedge \star(\zeta^{\rm de}(i_{\dot{V}} F) \wedge \widetilde{V}) + F \wedge \star(\zeta^{\rm de}(i_V F) \wedge \dot{\widetilde{V}})\\
&\quad + F \wedge \dot{\star}(\zeta^{\rm db}(i_V\star F) \wedge \widetilde{V}) + F \wedge \star(\zeta^{\rm db}(i_{\dot{V}}\star F) \wedge \widetilde{V}) + F \wedge \star(\zeta^{\rm db}(i_V\dot{\star}F) \wedge \widetilde{V})\\
&\quad + F \wedge \star(\zeta^{\rm db}(i_V\star F) \wedge \dot{\widetilde{V}}) + F \wedge \zeta^{\rm he}(i_{\dot{V}}F) \wedge \widetilde{V} + F \wedge \zeta^{\rm he}(i_V F) \wedge \dot{\widetilde{V}}\\
&\quad + F \wedge \zeta^{\rm hb}(i_{\dot{V}}\star F) \wedge \widetilde{V} + F \wedge \zeta^{\rm hb}(i_V\dot{\star}F) \wedge \widetilde{V} + F \wedge \zeta^{\rm hb}(i_V\star F) \wedge \dot{\widetilde{V}}
\end{aligned} \qquad (86)$$

where the subscript 0 is omitted on the right-hand side.

To determine the drive forms, observe that there are three different types of term in (86) which contain $\dot{\star}$, $\dot{V}$ or $\dot{\widetilde{V}}$. Since $\dot{\widetilde{V}} = (g(V))^{\cdot} = g(\dot{V}) + \dot{g}(V)$, terms in $\dot{\star}$, $\dot{V}$ and $\dot{g}(V)$ can be collected to give

$$2c\dot{\Lambda} = \dot{\Lambda}_{\dot{\star}} + \dot{\Lambda}_{\dot{V}} + \dot{\Lambda}_{\dot{g}(V)} \qquad (87)$$

where

$$2c\dot{\Lambda}_{\dot{\star}} = F\wedge\dot{\star}(\zeta^{\text{de}}(i_V F)\wedge\widetilde{V}) + F\wedge\dot{\star}(\zeta^{\text{db}}(i_V\star F)\wedge\widetilde{V})$$
$$+ F\wedge\star(\zeta^{\text{db}}(i_V\dot{\star}F)\wedge\widetilde{V}) + F\wedge\zeta^{\text{hb}}(i_V\dot{\star}F)\wedge\widetilde{V}, \tag{88}$$

$$2c\dot{\Lambda}_{\dot{V}} = F\wedge\star(\zeta^{\text{de}}(i_{\dot{V}}F)\wedge\widetilde{V}) + F\wedge\star(\zeta^{\text{de}}(i_V F)\wedge g(\dot{V}))$$
$$+ F\wedge\star(\zeta^{\text{db}}(i_{\dot{V}}\star F)\wedge\widetilde{V}) + F\wedge\star(\zeta^{\text{db}}(i_V\star F)\wedge g(\dot{V}))$$
$$+ F\wedge\zeta^{\text{he}}(i_{\dot{V}}F)\wedge\widetilde{V} + F\wedge\zeta^{\text{he}}(i_V F)\wedge g(\dot{V})$$
$$+ F\wedge\zeta^{\text{hb}}(i_{\dot{V}}\star F)\wedge\widetilde{V} + F\wedge\zeta^{\text{hb}}(i_V\star F)\wedge g(\dot{V}) \tag{89}$$

and

$$2c\dot{\Lambda}_{\dot{g}(V)} = F\wedge\star(\zeta^{\text{de}}(i_V F)\wedge\dot{g}(V)) + F\wedge\star(\zeta^{\text{db}}(i_V\star F)\wedge\dot{g}(V))$$
$$+ F\wedge\zeta^{\text{he}}(i_V F)\wedge\dot{g}(V) + F\wedge\zeta^{\text{hb}}(i_V\star F)\wedge\dot{g}(V). \tag{90}$$

The third term on the right-hand side of (88) may be expressed as

$$F\wedge\star(\zeta^{\text{db}}(i_V\dot{\star}F)\wedge\widetilde{V}) = F\wedge i_V\star(\zeta^{\text{db}}(i_V\dot{\star}F)) = -i_V F\wedge\star\zeta^{\text{db}}(i_V\dot{\star}F)$$
$$= i_V\dot{\star}F\wedge\star\zeta^{\text{he}}(i_V F) = -\dot{\star}F\wedge i_V\star\zeta^{\text{he}}(i_V F)$$
$$= -i_V\star\zeta^{\text{he}}(i_V F)\wedge\dot{\star}F = -F\wedge\dot{\star}\star(\zeta^{\text{he}}(i_V F)\wedge\widetilde{V})$$

using sequentially (3), (6), (44), (6), (1), (64). The fourth term on the right-hand side of (88) may be expressed as

$$F\wedge\zeta^{\text{hb}}(i_V\dot{\star}F)\wedge\widetilde{V} = -F\wedge\widetilde{V}\wedge\zeta^{\text{hb}}(i_V\dot{\star}F) = -F\wedge\widetilde{V}\wedge\star\star\zeta^{\text{hb}}(i_V\dot{\star}F)$$
$$= -\star\zeta^{\text{hb}}(i_V\dot{\star}F)\wedge\star(F\wedge\widetilde{V}) = i_V\star F\wedge\star\zeta^{\text{hb}}(i_V\dot{\star}F)$$
$$= i_V\dot{\star}F\wedge\star\zeta^{\text{hb}}(i_V\star F) = -\dot{\star}F\wedge i_V\star\zeta^{\text{hb}}(i_V\star F)$$
$$= -i_V\star\zeta^{\text{hb}}(i_V\star F)\wedge\dot{\star}F = -F\wedge\dot{\star}\star(\zeta^{\text{hb}}(i_V\star F)\wedge\widetilde{V})$$

using sequentially (1), (5), (64), (3), (44), (6), (1), (64).

Hence from (71)

$$2c\dot{\Lambda}_{\dot{\star}} = F\wedge\dot{\star}(\zeta^{\text{de}}(i_V F)\wedge\widetilde{V} + \zeta^{\text{db}}(i_V\star F)\wedge\widetilde{V} - \star(\zeta^{\text{he}}(i_V F)\wedge\widetilde{V}) - \star(\zeta^{\text{hb}}(i_V\star F)\wedge\widetilde{V}))$$
$$= F\wedge\dot{\star}G$$
$$= \dot{e}^a\wedge(F\wedge i_a\star G - i_a G\wedge\star F). \tag{91}$$

To collect terms in $\dot{\Lambda}_{\dot{V}}$ observe that by differentiating (73), $\dot{V} = \lambda V$ where $\lambda = \dot{e}^a(V)V_a$, one has

$$2c\dot{\Lambda}_{\dot{V}} = 2\lambda(F\wedge\star(\zeta^{\text{de}}(i_V F)\wedge\widetilde{V}) + F\wedge\star(\zeta^{\text{db}}(i_V\star F)\wedge\widetilde{V})$$
$$+ F\wedge\zeta^{\text{he}}(i_V F)\wedge\widetilde{V} + F\wedge\zeta^{\text{hb}}(i_V\star F)\wedge\widetilde{V})$$
$$= 2\lambda F\wedge\star G = 2i_V\dot{e}^a V_a F\wedge\star G = 2\dot{e}^a\wedge V_a i_V(F\wedge\star G).$$

The first two terms on the right-hand side of (90) become

$$F\wedge\star(\zeta^{\text{de}}(i_V F)\wedge\dot{g}(V)) + F\wedge\star(\zeta^{\text{db}}(i_V\star F)\wedge\dot{g}(V))$$
$$= F\wedge\star(i_V G\wedge\dot{g}(V)) = i_V G\wedge\dot{g}(V)\wedge\star F = -\dot{g}(V)\wedge i_V G\wedge\star F$$

and the last two terms on the right-hand side of (90) become

$$F\wedge\zeta^{\text{he}}(i_V F)\wedge\dot{g}(V) + F\wedge\zeta^{\text{hb}}(i_V\star F)\wedge\dot{g}(V) = -\dot{g}(V)\wedge F\wedge i_V\star G$$

so using $\dot{g}(V) = \dot{e}^a(V)e_a + \dot{e}^a e_a(V) = 2\dot{e}^a V_a + i_V(\dot{e}_a\wedge e^a)$ one has

$$2c\dot{\Lambda}_{\dot{g}(V)} = -\dot{g}(V)\wedge(i_V G\wedge\star F + F\wedge i_V\star G)$$
$$= -(2\dot{e}^a V_a + i_V(\dot{e}_a\wedge e^a))\wedge(i_V G\wedge\star F + F\wedge i_V\star G)$$
$$= -2\dot{e}^a\wedge(i_V G\wedge\star F + F\wedge i_V\star G) + \dot{e}_a\wedge e^a\wedge i_V(i_V G\wedge\star F + F\wedge i_V\star G)$$
$$= \dot{e}\wedge(-2(i_V G\wedge\star F + F\wedge i_V\star G) + e^a\wedge(i_V F\wedge i_V\star G - i_V G\wedge\star F)).$$

Adding this to $2\dot{\Lambda}_{\dot{V}}$ gives

$$
\begin{aligned}
2c\dot{\Lambda}_{\dot{V}} + 2c\dot{\Lambda}_{\dot{g}(V)} &= \dot{e}^a \wedge (2V_a(i_V F \wedge \star G - i_V G \wedge \star F) \\
&\quad + \dot{e}^a \wedge e^a \wedge (i_V F \wedge i_V \star G - i_V G \wedge \star F)) \\
&= 2V_a \dot{e}^a \wedge (i_V F \wedge \star G - i_V G \wedge \star F) - \dot{e}^a \wedge e^a \wedge i_V(i_V F \wedge \star G - i_V G \wedge \star F).
\end{aligned}
$$

Using the relation $\star G = i_V \star G \wedge \widetilde{V} + i_V \star i_V G$ and the similar relation for $\star F$, the combination above may be written as

$$
\begin{aligned}
&i_V F \wedge \star G - i_V G \wedge \star F \\
&\quad = i_V F \wedge i_V \star G \wedge \widetilde{V} + i_V F \wedge i_V \star i_V G - i_V G \wedge i_V \star F \wedge \widetilde{V} - i_V G \wedge i_V \star i_V F \\
&\quad = \star \mathbf{s} - i_V(i_V F \wedge \star i_V G - i_V G \wedge \star i_V F) = \star \mathbf{s}
\end{aligned}
$$

where the 1-form

$$\mathbf{s} = \star(i_V F \wedge i_V \star G \wedge \widetilde{V} + i_V \star F \wedge i_V G \wedge \widetilde{V}). \tag{92}$$

Hence,

$$2c\dot{\Lambda}_{\dot{V}} + 2c\dot{\Lambda}_{\dot{g}(V)} = \dot{e}^a \wedge (2V_a \star \mathbf{s} - e^a \wedge i_V \star \mathbf{s}). \tag{93}$$

Adding together (91) and (93) gives finally

$$2c\dot{\Lambda} = \dot{e}^a \wedge (F \wedge i_a \star G - i_a G \wedge \star F + 2V_a \star \mathbf{s} - e^a \wedge i_V \star \mathbf{s}). \tag{94}$$

Hence, the drive forms are given by

$$c\tau_a = \tfrac{1}{2}(F \wedge i_a \star G - i_a G \wedge \star F) + V_a \star \mathbf{s} - \tfrac{1}{2} e_a \wedge i_V \star \mathbf{s} \tag{95}$$

with associated stress–energy–momentum tensor

$$T = \tfrac{1}{2}(i_a F \otimes i^a G + i_a G \otimes i^a F - \star(F \wedge \star G) g + \widetilde{V} \otimes \mathbf{s} + \mathbf{s} \otimes \widetilde{V}). \tag{96}$$

The tensor T above coincides in Minkowski spacetime with that attributed historically to Abraham. It is derived here in a considerably wider context.

In terms of comoving fields, the drive forms can be written as

$$
\begin{aligned}
c\tau_a &= V_a i_V(\mathbf{e} \wedge \star \mathbf{d} + \mathbf{h} \wedge \star \mathbf{b}) - \tfrac{1}{2}(\mathbf{e} \wedge i_a \star \mathbf{d} + i_a \mathbf{d} \star \mathbf{e}) \\
&\quad - \tfrac{1}{2}(\mathbf{h} \wedge i_a \star \mathbf{b} + i_a \mathbf{b} \star \mathbf{h}) + 2v_a \mathbf{e} \wedge \mathbf{h} \wedge \widetilde{V} + e_a \wedge \mathbf{e} \wedge \mathbf{h}
\end{aligned} \tag{97}
$$

and hence

$$
\begin{aligned}
T &= -\tfrac{1}{2}(\mathbf{e} \otimes \mathbf{d} + \mathbf{d} \otimes \mathbf{e}) - \tfrac{1}{2}(\mathbf{h} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{h}) \\
&\quad + \tfrac{1}{2}(g(\tilde{\mathbf{e}}, \tilde{\mathbf{d}}) + g(\tilde{\mathbf{h}}, \tilde{\mathbf{b}}))(g + 2\widetilde{V} \otimes \widetilde{V}) + (\widetilde{V} \otimes \tilde{S} + \tilde{S} \otimes \widetilde{V})
\end{aligned} \tag{98}
$$

where the Poynting 1-form

$$\tilde{S} = \star(\widetilde{V} \wedge \mathbf{e} \wedge \mathbf{h}).$$

One may express the expressions above in terms of comoving polarization 1-forms **p** and magnetization 1-forms **m**, defined in terms of comoving electromagnetic fields by

$$\mathbf{d} = \mathbf{e} + \mathbf{p} \tag{99}$$

$$\mathbf{h} = \mathbf{b} - \mathbf{m}. \tag{100}$$

Thus,

$$G = F + \mathcal{P} \tag{101}$$

where

$$\mathcal{P} = \mathbf{p} \wedge \tilde{V} + \star(\mathbf{m} \wedge \tilde{V}). \tag{102}$$

Then, one finds

$$\tau_c = \tau_c^1 + \tau_c^2 + \tau_c^3 + \tau_c^4$$

where

$$2c\tau_c^1 = i_c \star G \wedge F - i_c G \wedge \star F,$$
$$2c\tau_c^2 = V_c(\mathbf{p} \wedge \star F + \mathbf{m} \wedge F - (\mathbf{p} \wedge \mathbf{b} + \mathbf{m} \wedge \mathbf{e}) \wedge \tilde{V})$$
$$2c\tau_c^3 = -(i_c F) \wedge (\mathbf{m} \wedge \tilde{V}) - (i_c \star F) \wedge \star(\mathbf{m} \wedge \tilde{V})$$
$$2c\tau_c^4 = -V_c(\mathbf{p} \wedge \star F + \mathbf{m} \wedge F) - V_c \tilde{V} \wedge (\mathbf{p} \wedge \mathbf{b} + \mathbf{m} \wedge \mathbf{e}) - e_c \wedge (\mathbf{p} \wedge \mathbf{b} + \mathbf{m} \wedge \mathbf{e}).$$

The above are valid for all simple media in arbitrary gravitational fields. For a simple medium, which may be inhomogeneous, anisotropic and intrinsically magneto-electric, at rest in an inertial frame in *Minkowski spacetime* with Minkowski coordinates $\{t, \vec{x}\}$ one has (in Euclidean notation)

$$\widetilde{V} = -\mathrm{d}t, \qquad \mathbf{e} = \vec{E} \cdot \mathrm{d}\vec{x}, \qquad \mathbf{b} = \vec{B} \cdot \mathrm{d}\vec{x}, \qquad \mathbf{h} = \vec{H} \cdot \mathrm{d}\vec{x}, \qquad \mathbf{d} = \vec{D} \cdot \mathrm{d}\vec{x}$$

and

$$g(\tilde{\mathbf{e}}, \tilde{\mathbf{d}}) = \vec{E} \cdot \vec{D}, \qquad g(\tilde{\mathbf{h}}, \tilde{\mathbf{b}}) = \vec{H} \cdot \vec{B}$$

$$\tilde{S} = -(\vec{E} \times \vec{H}) \cdot \mathrm{d}\vec{x}.$$

The coordinate components of the stress–energy–momentum tensor follow as

$$\begin{aligned} T_{00} &= \tfrac{1}{2}(\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B}) \\ T_{ij} &= -\tfrac{1}{2}(E_i D_j + E_j D_i) - \tfrac{1}{2}(H_i B_j + B_j H_i) + \tfrac{1}{2}\delta_{ij}(\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B}) \\ T_{0k} &= T_{k0} = -(\vec{E} \times \vec{H})_k. \end{aligned} \tag{103}$$

11. Conclusions

Natural assumptions made above for the dependence of the constitutive tensor Z on the normalized 4-velocity of a simple medium have led via a non-trivial variational argument to a contribution to the stress–energy–momentum tensor (involving phenomenological electromagnetic interactions with bulk matter) that coincides with that suggested by Abraham under more restricted circumstances. Although natural, the assumptions based on physical considerations are not, however, necessarily the simplest to make.

If Z is chosen to be independent of the metric and hence $\tilde{V}$, with $Z_t = Z_0$ and $\dot{Z} = 0$ so that $G = Z_0(F)$ in all gravitational fields, one obtains immediately from the above variational calculations (72) the drive forms

$$c\tau_a = \tfrac{1}{2}(F \wedge i_a \star G - i_a G \wedge \star F) \tag{104}$$

and the associated stress–energy–momentum tensor

$$T = \tfrac{1}{2} i_a G \otimes i^a F + \tfrac{1}{2} i_a F \otimes i^a G - \tfrac{1}{2} \star (F \wedge \star G) g \tag{105}$$

showing clearly its independence of the 4-velocity of the medium. It is of interest to note that such a tensor coincides with that obtained by symmetrizing the one proposed by Minkowski.

In the absence of a generally accepted relativistic covariant description of deformable matter interacting with electromagnetic fields, the adoption of a particular stress–energy–momentum tensor for the electromagnetic field alone in polarizable (and possible magneto-electric) media must remain a matter of expediency. However, useful models for the total stress–energy–momentum tensor for such systems can benefit from the use of sufficiently general phenomenological descriptions of the electromagnetic properties of moving media compatible with relativistic covariance. For example, a thermodynamically inert (pressureless, cold) fluid can be modelled by adding the electromagnetic stress–energy–momentum tensor (96) to the matter stress–energy–momentum tensor $\frac{m_0}{c\epsilon_0}\mathcal{N}\widetilde{V}\otimes\widetilde{V}$ where $\mathcal{N}$ is a scalar number density field, m_0 some constant with the dimensions of mass and V the unit timelike 4-velocity field of the fluid. Supplemented with continuity conditions, the vanishing divergence of such a combination yields the dynamics of the system and with prescribed boundary conditions at an interface separating such media with different properties one may compute bulk forces and torques.

A review has also been given of the symmetry constraints expected of the total stress–energy–momentum tensor particularly when this is considered to be a source of relativistic gravitation. This led to a definition in terms of a variational derivative and a consideration of the response of the electromagnetic constitutive properties to gravitational perturbations. It is suggested that stress–energy–momentum tensors parameterized by a self-adjoint constitutive tensor Z offer a viable means to explore the electromagnetic properties of a range of inhomogeneous, anisotropic and possibly magneto-electric continua, at least in regions where dispersion and losses can be ignored to a first approximation. This formulation suggests a method to determine the properties of Z by exploring its phenomenological response to electromagnetic fields in arbitrarily moving reference frames and variable gravitational fields. It opens up the possibility of performing such experiments in new environments such as those carried out under terrestrial free-fall or space station situations or in astrophysical contexts.

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Appendix

Using the notation established in the text, this appendix derives the useful formula (63) relating $(\star\Psi)\dot{}$ to $\star\dot{\Psi}$ where $\Psi\in\Gamma\Lambda^p M$. Let I denote a multi-index constructed from the single indices $a, b, c\ldots$ in the range 0, 1, 2, 3 where the components of the metric tensor g and Ψ in an g-orthonormal basis $\{e^c\}$ are, respectively, η_{ab} and Ψ_I. Thus,

$$\Psi = \Psi_I e^I$$

and

$$\dot{\Psi} = \dot{\Psi}_I e^I + \Psi_I (e^I)\dot{}. \tag{A.1}$$

Since e^I is the exterior product of p 1-forms

$$(e^I)\dot{} = \dot{e}^c \wedge i_c(e^I).$$

Similarly, since the basis is orthonormal

$$(\star e^I)\dot{} = \dot{e}^c \wedge i_c(\star e^I).$$

Thus, using (A.1)

$$\dot{\Psi}_I e^I = \dot{\Psi} - \dot{e}^c \wedge i_c\Psi.$$

Applying $\star$ to this gives

$$\dot{\Psi}_I \star e^I = \star\dot{\Psi} - \star(\dot{e}^c \wedge i_c\Psi). \tag{A.2}$$

But

$$
\begin{aligned}
(\star\Psi)^{\cdot} &= (\Psi_I \star e^I)^{\cdot} \\
&= \dot{\Psi}_I \star e^I + \Psi_I (\star e^I)^{\cdot} \\
&= \dot{\Psi}_I \star e^I + \Psi_I \dot{e}^c \wedge i_c (\star e^I) \\
&= \dot{\Psi}_I \star e^I + \dot{e}^c \wedge i_c (\star \Psi).
\end{aligned}
$$

Substituting from (A.2) yields the relation

$$(\star\Psi)^{\cdot} = \dot{e}^c \wedge i_c(\star\Psi) - \star(\dot{e}^c \wedge i_c\Psi) + \star\dot{\Psi}.$$

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